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Tangency and Differentiation: Marginal Functions

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INTRODUCTION

Let f be an extended-real-valued function on a product of two sets $X \times Y$. The *marginal function* $\inf_X f$ associated with f (also called the *value function*) is defined on Y by

$$\inf_X f(y) = \inf_{x \in X} f(x, y). \tag{0.1}$$

The behavior of $\inf_X f$ determines the range of penalty and duality methods which are valid for the problem of minimizing $f(\cdot, y)$ on X for a given parameter y. In particular, various unilateral (directional) derivatives of the marginal function furnish "measures of sensitivity"' of the minimization problems. They have an intimate relationship to the Lagrange multipliers feasible for the corresponding minimization problems [35, 19, 33, ...].

In this paper we deal with upper and lower estimates of two such derivatives (*contingent* and *equi-tangent*) of $\inf_X f$ in terms of the corresponding derivatives of f. Further specializations will be given in the case of $f = g \dotplus \psi_A$, where $g\colon X \times Y \to \overline{\mathbb{R}}$, $A \subset X \times Y$ represents the explicit-constraints relation, ψ_A is the indicator function of A and $\dotplus$ is the upper extension of the addition.

The contingent (epi-) derivative at z of a function g on a quasi-topological vector space Z is the following (positively homogeneous, lower semicontinuous) functions $D_{(-;-)}g(z)\colon Z \to \overline{\mathbb{R}}$:

$$D_{(-;-)}g(z)h = \sup_{Q \in \mathcal{N}(h)} \sup_{t>0} \inf_{0<t'<t} \inf_{h' \in Q} \frac{1}{t'}[g(z + t'h') - g(z)].$$

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Slightly generalizing a result of Penot [32] we have that

$$D_{(-;\,-)}\inf_X f(y)k \le \inf_{h\in X} D_{(-;\,-)}f(x,y)(h,k) \tag{0.2}$$

for each $x \in \mathrm{Min}_X f(y) = \{x' \in X\colon f(x',y) \le \inf_X f(y)\}$. (If f is of the form $g \dotplus \psi_A$, we shall denote by $\inf_A g$ and $\mathrm{Min}_A g$ the corresponding marginal function and the multifunction of minima.)

In order to actually achieve the equality in (0.2) we used some weak compactness-like properties of the families of approximate minima. In particular, the equality holds in (0.2), whenever the multifunction

$$(t',k') \mapsto \frac{1}{t'}\left[\mathrm{Min}_X f(y+t'k') - x\right]$$

has nonempty upper limit (adherence) as t' tends to 0 and k' to k.

Penot establishes the equality in (0.2) for Banach spaces in the case where $D_{(-;\,-)}f(x,y)(h,0)$ is strictly positive for $h \neq 0$ [32]. We extend his result to locally convex spaces.

The equi-tangent derivative (usually called *generalized Clarke derivative*) $D_{(+,\,+;\,-)}g$ of a lower semicontinuous function g is defined by

$$D_{(+,\,+;\,-)}g(z)h = \sup_{Q\in\mathscr{N}(h)} \inf_{\substack{W\in\mathscr{N}(z)\\ t>0}} \sup_{\substack{z'\in W\\ t'<t}} \inf_{h'\in Q} \frac{1}{t'}\left(f(z'+t'h') - g(z')\right),$$

where $\mathscr{N}(z)$ is the neighborhood filter of z for the supremum of the considered topology of Z and of the weakest topology for which g is upper semicontinuous [38, 37, 10]. The study of estimates for the equitangent derivative of marginal functions has been initiated by Clarke [3] and continued by numerous authors [15, 19, 33]... .

In a series of papers [42, 39, 36, 35], Rockafellar extends to non locally Lipschitzian functions a theorem of Hiriart–Urruty [23] to the effect that

$$D_{(+,\,+;\,-)}\inf_X f(y)k \le \sup_{x\in\mathrm{Min}_X f(y)} \inf_{h\in X} D_{(+,\,+;\,-)}f(x,y)(h,k), \tag{0.3}$$

where the equi-tangent derivatives are calculated with respect to the usual topologies in finite dimensional spaces, f is supposed to be lower semicontinuous and *tame* at y. The latter property is equivalent to the existence of a compact set K in X and a neighborhood W of y (in the supremum of the usual topology and the coarsest topology for which $\inf_X f$ is upper semicontinuous) such that $\mathrm{Min}_X f(y')$ meets K for each $y' \in W$.

We establish (0.3) for a broad class of topologies (that contains linear topologies) removing the restriction of finite dimension. The compactness-like properties that we use are also considerably weaker than tameness: $\mathrm{Min}_X f(y')$ may escape from any bounded set rolling over directions as y' tends to y.

This sensible improvement of existent results is due to the use of the theory of Γ-limits (originated by De Giorgi and Franzoni [6]) which provides insight and versatile tools (e.g., [5, 21, 22]). This paper is intended as an illustration of some applications of that theory.

Theory of compactoid and compact filters [13, 11, 30] finds here its consecutive application. Results on lower semicontinuity of marginal functions (e.g., [11]) are applied in this paper to difference quotients. We use as well some stability results of the relation of minima $\mathrm{Min}_X f$ (e.g., [12]).

1. Generalized Derivatives

We are concerned with unilateral directional derivatives, i.e., with positively homogeneous functions which epigraphs approximate the epigraphs of given functions. It was observed [10] that the unilateral derivatives encountered in the literature may be expressed as appropriate Γ-functionals of difference-quotients.

For $1 \leq i \leq n$, let X_i be a nonempty set, $\mathscr{F}_i$ a filter on X_i and α_i a sign ($+$ or $-$). By using the convention that $\mathrm{ext}^+ = \sup$ and $\mathrm{ext}^- = \inf$, we define, after De Giorgi [5], Γ-functionals of functions $f\colon \prod_{i=1}^n X_i \to \overline{\mathbb{R}}$:

$$\lim_{\Gamma(\mathscr{F}_1^{\alpha_1}, \ldots, \mathscr{F}_n^{\alpha_n})} f = \mathop{\mathrm{ext}^{-\alpha_n}}_{F_n \in \mathscr{F}_n} \ldots \mathop{\mathrm{ext}^{-\alpha_1}}_{F_1 \in \mathscr{F}_1} \mathop{\mathrm{ext}^{\alpha_1}}_{x_1 \in F_1} \ldots \mathop{\mathrm{ext}^{\alpha_n}}_{x_n \in F_n} f(x_1, \ldots, x_n). \tag{1.1}$$

If, for some i, $1 \leq i \leq n$, τ_i is a topology on X_i, then we write

$$\left(\lim_{\Gamma(\mathscr{F}_1^{\alpha_1}, \ldots, \tau_i^{\alpha_i}, \ldots, \mathscr{F}_n^{\alpha_n})} f\right)(x_i) \quad \text{for} \quad \lim_{\Gamma(\mathscr{F}_1^{\alpha_1}, \ldots, \mathscr{N}_{\tau_i}(x_i)^{\alpha_i}, \ldots, \mathscr{F}_n^{\alpha_n})} f, \tag{1.2}$$

where $\mathscr{N}_{\tau_i}(x_i)$ is the neighborhood filter at x_i for τ_i. For instance, let τ be a topology on a set X and let $\mathbf{f} = \{f_t\}_{t \in T}$ be a family of functions on X filtered by $\mathscr{F}$ on T. Then, on putting $\mathbf{f}(t, x) = f_t(x)$, one obtains

$$\begin{aligned} \left(\lim_{\Gamma(\mathscr{F}^+, \tau^-)} \mathbf{f}\right)(x) &= \sup_{Q \in \mathscr{N}_\tau(x)} \inf_{F \in \mathscr{F}} \sup_{t \in F} \inf_{x' \in Q} f_t(x') \\ \left(\lim_{\Gamma(\mathscr{F}^-, \tau^-)} \mathbf{f}\right)(x) &= \sup_{Q \in \mathscr{N}_\tau(x)} \sup_{F \in \mathscr{F}} \inf_{t \in F} \inf_{x' \in Q} f_t(x'), \end{aligned} \tag{1.3}$$

respectively, the *upper* and *lower epi-limits* (also called *variational limits* or *infimal limits*) (e.g., [5, 6, 10, 21]). They are frequently denoted by $ls_{\mathscr{F}}^\tau$ and $li_{\mathscr{F}}^\tau$ ($\limsup_{\mathscr{F}}^\tau$, $\liminf_{\mathscr{F}}^\tau$). In particular, $\Gamma(\tau^-)f = \mathrm{cl}_\tau f$, the τ-lower semicontinuous hull of f.

When we need to indicate the link between variables and extremizations we use the convention where $x' \to^{\tau^\alpha} x$ substitutes $N_\tau(x)^\alpha$. For example, given topologies τ on X and σ on Y and a function $f\colon X \times Y \to \overline{\mathbb{R}}$, we write

$$\lim{}_{\Gamma(x' \xrightarrow{\tau^+} x,\, y' \xrightarrow{\sigma^-} y)} f(x', y') \qquad \text{for} \quad \left(\lim{}_{\Gamma(\tau^+, \sigma^-)} f\right)(x, y).$$

If a topology in question is fixed or indicated by the context, we omit its name in the above formula.

A topology on a linear space is called *quasi-linear* [10], if the addition is continuous and, if for each $\lambda \in \mathbb{R}$, the mapping $x \mapsto \lambda x$ is continuous. Let τ be a quasi-linear topology on X. Consider an extended-real-valued function f on X, finite at x. The *contingent*, *tangent*, and *interior* (*epi-*) *derivatives* of f at x are, respectively,

$$D_{(-;\tau^-)} f(x)h = \lim{}_{\Gamma(t \xrightarrow{-} 0,\, h' \xrightarrow{\tau^-} h)} \frac{1}{t}[f(x + th') - f(x)] \tag{1.4}$$

$$D_{(+;\tau^-)} f(x)h = \lim{}_{\Gamma(t \xrightarrow{+} 0,\, h' \xrightarrow{\tau^-} h)} \frac{1}{t}[f(x + th') - f(x)] \tag{1.5}$$

$$D_{(+;\tau^+)} f(x)h = \lim{}_{\Gamma(t \xrightarrow{+} 0,\, h' \xrightarrow{\tau^+} h)} \frac{1}{t}[f(x + th') - f(x)]. \tag{1.6}$$

Classical *Dini* (directional) derivatives constitute a special case of the above derivatives. When the topology on X is fixed we shall also write $D_{(-;\,-)}, D_{(+;\,-)}, D_{(+;\,+)}$.

Consider another topology σ on X. We define the *equi-tangent* (*epi-*) *derivative* of f (in fact, σ-equi τ-tangent derivative) by

$$\begin{aligned} D_{(+,\sigma^+;\tau^-)} f(x)h = {} & \lim{}_{\Gamma(t \xrightarrow{+} 0,\, (x', r') \xrightarrow{(\sigma \times \nu) \vee \operatorname{epi} f^+} (x, f(x)),\, h' \xrightarrow{\tau^-} h)} \\ & \frac{1}{t}[f(x' + th') - r'], \end{aligned} \tag{1.7}$$

where $(\sigma \times \nu) \vee \operatorname{epi} f$ is the restriction to the epigraph of f of the product of σ and of the usual topology ν of the real line. Similarly, we define the *equi-interior* (*epi-*) *derivative* (more precisely, σ-equi τ-interior derivative) $D_{(+,\sigma^+;\tau^+)} f(x)h$.

We denote by ι the discrete topology on X (which is quasi-linear). The above-defined derivatives with respect to ι: $D_{(-;\iota^-)}$, $D_{(+;\iota^-)} = D_{(+;\iota^+)}$, $D_{(+,\sigma^+;\iota^-)} = D_{(+,\sigma^+;\iota^+)}$ are said to be *radial*. Note that in the case of the discrete topology (as the "second" topology) the equi-tangent derivative coincides with the tangent derivative: $D_{(+,\iota^+;\tau^-)} = D_{(+;\tau^-)}$ and, similarly, $D_{(+,\iota^+;\tau^+)} = D_{(+;\tau^+)}$. The above observation justifies the name "equi-tangent derivative." It is also called the *(generalized) Clarke (directional) derivative* in honor of F. H. Clarke who introduced the concept in [3] in the case of locally Lipschitzian functions on the Euclidean space. Its extension (1.7) is due to R. T. Rockafellar (e.g., [37, 38], ...]) for the cases $\tau = \sigma$ and $\tau = \iota$. Let $\sigma_{\text{epi} f} = \sigma \vee (\text{epi } f)^{-1}\nu$, where $(\text{epi } f)^{-1}\nu$ is the coarsest topology on X for which f is upper semicontinuous. Then $D_{(+,\sigma_f^+;\tau^-)}f = D_{(+,\sigma^+;\tau^-)}f$. Finally, if f is lower semicontinuous at x in σ, then it is known [37, 24, 10] that

$$D_{(+,\sigma^+;\tau^-)}f(x)h = \lim_{\Gamma(t \xrightarrow{+} 0,\, x' \xrightarrow{\sigma \text{epi} f^+} x,\, h' \xrightarrow{\tau} h)} \frac{1}{t}\left[f(x'+th') - f(x')\right]. \tag{1.8}$$

If $\tau \geq \sigma$ and σ is a linear topology, then $D_{(+,\sigma^+;\tau^-)}f(x)$ and $D_{(+,\sigma^+;\tau^+)}f(x)$ are convex positively homogeneous [38, 24, 10].

2. Approximating Cones

The *indicator function* ψ_A of a subset A of a set X is defined

$$\psi_A(x) = \begin{cases} 0 & \text{if } x \in A \\ +\infty, & \text{otherwise.} \end{cases}$$

Γ-functionals applied to indicator functions (or to characteristic functions of sets) have been used to define various limits of filtered families of sets (e.g., [5, 10, 21]). In particular, the classical upper and lower limits of filtered families of sets have been recovered in this way.

Suppose that, for $1 \leq i_1 \leq \cdots \leq i_m \leq n$, τ_{i_l} is a topology on X_{i_l}, while for the remaining indices i, we consider filters $\mathscr{F}_i$ on X_i. Let $\mathbf{A}$ be a subset of $X_1 \times \cdots \times X_n$. Then $\operatorname{Lim}_{(\mathscr{F}_1^{\alpha_1},\ldots,\tau_{i_1}^{\alpha_{i_1}},\ldots,\mathscr{F}_n^{\alpha_n})} \mathbf{A}$ is the subset of $X_{i_1} \times \cdots \times X_{i_m}$ defined by

$$\psi_{\operatorname{Lim}(\mathscr{F}_1^{\alpha_1},\ldots,\tau_{i_1}^{\alpha_{i_1}},\ldots)\mathbf{A}} = \lim_{\Gamma(\mathscr{F}_1^{\alpha_1},\ldots,\tau_{i_1}^{\alpha_{i_1}},\ldots)} \psi_{\mathbf{A}}. \tag{2.1}$$

For example, take $\mathbf{A} = \{A_t\}_{t \in T}$, where, for each $t \in T$, $A_t \subset X$. In other words, $\mathbf{A}$ is a subset of $T \times X$. Let $\mathscr{F}$ be a filter on T, τ a topology on X. Then $x \in \operatorname{Lim}_{(\mathscr{F}^-,\tau^-)} \mathbf{A}$ (frequently denoted by $Ls^{\tau}_{\mathscr{F}}\mathbf{A}$), if and only if for every $Q \in \mathscr{N}_\tau(x)$, each $F \in \mathscr{F}$ there is $t \in F$ such that $A_t \cap Q \neq \varnothing$, while, $x \in \operatorname{Lim}_{(\mathscr{F}^+,\tau^-)} \mathbf{A}$ (frequently denoted by $Li^{\tau}_{\mathscr{F}}\mathbf{A}$), if and only if for each $Q \in \mathscr{N}_\tau(x)$ there exists $F \in \mathscr{F}$ such that, for every $t \in F$, $A_t \cap Q \neq \varnothing$.

Disposing with the notions of limit (2.1), we are in a position to define various cones that locally approximate subsets of linear spaces. Many, more or less classical, approximating cones may be so introduced. While dealing with subsets C of a real linear space Y, we consider the *translation-homothety* (the counterpart of the difference quotient), which is the following relation (multifunction) from $Y \times \mathbb{R}^+$ to Y:

$$(y, t) \mapsto \frac{1}{t}(C - y).$$

One notes that the difference quotient of the indicator function of C is equal to the indicator function of the translation-homothety of C (i.e., of the set $\{(y, t, h)\colon\ h \in (1/t)(C - y)\}$. For this reason, every derivative defined in Section 1 has its counterpart in an approximating cone. Let θ be a quasi-linear topology on Y, C a (nonempty) subset of Y. The *contingent*, *tangent*, and *interior cones* of C at y are defined, respectively, by

$$T_{(-;\theta^-)}C(y) = \mathrm{Lim}_{(t \overset{-}{\to} 0, \theta^-)} \frac{1}{t}(C - y) \tag{2.2}$$

$$T_{(+;\theta^-)}C(y) = \mathrm{Lim}_{(t \overset{+}{\to} 0, \theta^-)} \frac{1}{t}(C - y) \tag{2.3}$$

$$T_{(+;\theta^+)}C(y) = \mathrm{Lim}_{(t \overset{+}{\to} 0, \theta^+)} \frac{1}{t}(C - y). \tag{2.4}$$

When ρ is another topology on Y, then the *equi-tangent* (i.e., ρ-equi θ-tangent) *cone* and the *equi-interior cone* of C at y are defined by

$$T_{(+,\rho^+;\theta^-)}C(y) = \mathrm{Lim}_{(t \overset{+}{\to} 0,\, y' \xrightarrow{(\rho \vee C)^+} y,\, \theta^-)} \frac{1}{t}(C - y') \tag{2.5}$$

$$T_{(+,\rho^+;\theta^+)}C(y) = \mathrm{Lim}_{(t \overset{+}{\to} 0,\, y' \xrightarrow{(\rho \vee C)^+} y,\, \theta^+)} \frac{1}{t}(C - y'), \tag{2.6}$$

where $\rho \vee C$ stands for the restriction of the topology ρ to the set C.

Applying the limits (2.1) to graphs, epigraphs, and hypographs of functions, one obtains various notions of limits of filtered families of functions. Some of them have been already defined directly in Section 1, for instance, one has [5, 6, 10]

$$\mathrm{epi}\big(\lim{}_{\Gamma(\mathscr{F}^+, \tau^-)} \mathbf{f}\big) = \mathrm{Lim}_{(\mathscr{F}^+, \tau \times \nu^-)} \exp \mathbf{f}$$

$$\mathrm{epi}\big(\lim{}_{\Gamma(\mathscr{F}^-, \tau^-)} \mathbf{f}\big) = \mathrm{Lim}_{(\mathscr{F}^-, \tau \times \nu^-)} \mathrm{epi}\, \mathbf{f}$$

and also [10]

$$\operatorname{epi}(\lim_{\Gamma(\mathscr{F}^+,\tau^+)}\mathbf{f}) = \operatorname{Lim}_{(\mathscr{F}^+,\tau^+,\nu^-)}\operatorname{epi}\mathbf{f}.$$

(We recall that ν stands for the usual topology of the real line.)

This possibility of passing from the limits of families of sets to the corresponding (epi-) limits of families of functions and the fact that

$$\frac{1}{t}[\operatorname{epi} f - (x, r)] = \operatorname{epi}\left[\frac{1}{t}f(x + t\,\cdot) - r\right] \tag{2.7}$$

enables us to see the epi-derivatives (1.3) through (1.6) as approximating conesto eipgraphs.

Namely,

$$\operatorname{epi}\bigl(D_{(-;\tau^-)}f(x)\bigr) = \bigl(T_{(-;\tau\times\nu^-)}\operatorname{epi} f\bigr)(x, f(x)) \tag{2.8}$$

$$\operatorname{epi}\bigl(D_{(+;\tau^-)}f(x)\bigr) = \bigl(T_{(+;\tau\times\nu^-)}\operatorname{epi} f\bigr)(x, f(x)) \tag{2.9}$$

$$\operatorname{epi}\bigl(D_{(+,\sigma^+;\tau^-)}f(x)\bigr) = \bigl(T_{(+,\sigma\times\nu^+;\tau\times\nu^-)}\operatorname{epi} f\bigr)(x, f(x)) \tag{2.10}$$

$$\operatorname{epi}\bigl(D_{(+;\sigma^+)}f(x)\bigr) = \bigl(T_{(+;\sigma^+,\nu^-)}\operatorname{epi} f\bigr)(x, f(x)) \tag{2.11}$$

$$\operatorname{epi}\bigl(D_{(+,\sigma^+;\tau^+)}f(x)\bigr) = \bigl(T_{(+,\sigma\times\nu^+;\tau^+,\nu^-)}\operatorname{epi} f\bigr)(x, f(x)). \tag{2.12}$$

It is known [24, 38, 44, 10] under some mild assumptions on topologies involved that the equi-tangent cone is convex. A similar argument proves the following:

THEOREM 2.1. *Let θ be a quasi-linear topology finer than a linear topology ρ. If $T_{(+,\rho^+;\theta^+)}C(y)$ is nonempty then*

$$T_{(+,\rho^+;\theta^+)}C(y) + T_{(+,\rho^+;\theta^-)}C(y) \subset T_{(+,\rho^+;\theta^+)}C(y). \tag{2.13}$$

This theorem implies the above-mentioned convexity of equi-tangent and equi-interior cones. Since $T_{(+,\rho^+;\theta^+)}C(y)$ is open with respect to θ, it is equal to the interior of $T_{(+,\rho^+;\theta^-)}C(y)$ provided that θ is linear [26]. Therefore one actually has this generalization of [32]:

COROLLARY 2.2. *Let θ, ρ be linear topologies and $\theta \geq \rho$. If $T_{(+,\rho^+;\theta^+)}C(y)$ is nonempty, then*

$$\operatorname{cl}_\theta\bigl(T_{(+,\rho^+;\theta^+)}C(y)\bigr) = \operatorname{cl}_\theta\bigl(T_{(+,\rho^+;\iota^-)}C(y)\bigr) = T_{(+,\rho^+;\theta^-)}C(y). \tag{2.14}$$

COROLLARY 2.3. *Suppose that θ and ρ are linear topologies and $\theta \geq \rho$. If $T_{(+,\rho^+;\theta^+)}C(y)$ is nonempty, $y \in C$, then*

$$\operatorname{Lim}_{(y' \xrightarrow{(\rho \vee C)^+} y,\, \theta^-)} T_{(+,\rho^+;\theta^-)}C(y') = T_{(+,\rho^+;\theta^-)}C(y). \tag{2.15}$$

Proof. Since $y \in C$, we have the inclusion $\subset$. On the other hand, one easily checks that

$$\operatorname{Lim}_{(y' \xrightarrow{(\rho \vee C)^+} y, \theta^-)} T_{(+,\rho^+;\theta^-)} C(y') \supset \operatorname{Lim}_{(y' \xrightarrow{(\rho \vee C)^+} y, \iota^-)} T_{(+,\rho^+;\iota^-)} C(y')$$
$$\supset T_{(+,\rho^+;\iota^-)} C(y). \tag{2.16}$$

Now apply cl_θ to both the sides and recall Corollary 2.2. □

3. Hypocompactoid Relations

Some types of results concerning differentiation of marginal functions that we are going to present require certain compactness-like assumptions. Traditional notions of compactness are unnecessarily stringent for our purposes and may be substituted by weaker properties [23, 36, 31, 11].

A filter (or a filter base) $\mathscr{F}$ on a topological space is called *compactoid* if every ultrafilter $\mathscr{U}$ finer than $\mathscr{F}$ ($\mathscr{U} \in \beta(\mathscr{F})$) is convergent. In particular, a nonempty set A is *compactoid* if its principal filter is compactoid (every ultrafilter containing A converges).

Consider a relation $M \subset Y \times X$, where X is a topological space and $\mathscr{G}$ a filter on Y. Suppose that $\mathscr{G}$ meets the domain of M, i.e., the set $M^{-1}X = \{y \in Y\colon M(y) \neq \varnothing\}$. M is called *compactoid along* $\mathscr{G}$, if the filter generated by $M\mathscr{G} = \{MG\colon G \in \mathscr{G}\}$ is compactoid.

A mapping m from a subset of Y to X is called a $\mathscr{G}$*-eventual selection* of M, if there exists $G \in \mathscr{G}$ such that $m(y) \in M(y)$ for each $y \in G$.

A relation $M\colon Y \overset{\scriptscriptstyle\wedge}{<} X$ is said to be *hypocompactoid* along a filter $\mathscr{G}$, if for every ultrafilter $\hat{\mathscr{U}}$ finer than (MG and such that $M^{-1}X = \{y \in Y\colon M(y) \neq \varnothing\} \in \mathscr{U}$, there exists a $\mathscr{U}$-eventual selection m of M, for which $m(\mathscr{U})$ converges. In particular, if A is a compactoid set (for example, a relatively compact set) such that $M(y) \cap A \neq \varnothing$ for some $G \in \mathscr{G}$ and each $y \in G$, then M is hypocompactoid along $\mathscr{G}$.

In particular, if M has a $\mathscr{G}$-eventual selection m for which $m(\mathscr{G})$ is compactoid, then M is hypocompactoid along $\mathscr{G}$.

A relation from a topological space (Y, θ) to a topological space (X, τ), hypocompactoid along a neighbourhood filter $\mathscr{N}_\theta(y)$ is called (θ*-locally*) τ*-hypocompactoid* at y.

The above notion of hypocompactoidness at y is weaker than a concept introduced by Penot [31, Definition 1.3] as a variant of lower semicontinuity.

A filter $\mathscr{F}$ on a topological space is called *sequentially compactoid* if every sequence finer than $\mathscr{F}$ (i.e., such that the elementary filter it generates is finer than $\mathscr{F}$) admits a convergent subsequence. A countably based filter in a first countable space is sequentially compactoid, if and only if, it is compactoid. A relation M: $Y \rightrightarrows X$ is called sequentially hypocompactoid, if for every sequence (y_n) on $M^{-1}X$, there exists a subsequence (y_{n_k}) and a convergent sequence (x_k) such that $x_k \in My_{n_k}$.

Sequentially hypocompactoid and compactoid relations are not comparable in general.

In the sequel we shall use a special case of Theorem 6.2 of [11]. Recall that the sets of *ε-approximate minima* of f are given for each $\varepsilon > 0$ by:

$$^{\varepsilon}\mathrm{Min}_X f(y) = \begin{cases} \{x: f(x,y) \le -1/\varepsilon\} & \text{if } \inf_X f(y) = -\infty, \\ \{x: f(x,y) \le \inf_X f(y) + \varepsilon\} & \text{otherwise.} \end{cases}$$

THEOREM 3.1. *Let* f: $X \times Y \to \overline{\mathbb{R}}$ *be lower semicontinuous on* $X \times \{y\}$ *and let, for every small* $\varepsilon > 0$, $^{\varepsilon}\mathrm{Min}_X f$ *be hypocompactoid at* y. *Then* $\inf_X f$ *is lower semicontinuous at* y.

Proof. Suppose that $\inf_X f$ is not lower semicontinuous at y: there exists $s < r < \inf_X f(y)$ and an ultrafilter $\mathscr{U}$ convergent to y such that $\inf_X f(y') < s$, for each y' in some $U \in \mathscr{U}$. For $\varepsilon = \min(r - s, |1/r|)$, $^{\varepsilon}\mathrm{Min}_X f(y') \subset \{x': f(x', y') \le r\}$. By hypocompactoidness, there is a selection m: $U \to X$ of $^{\varepsilon}\mathrm{Min}_X f$ such that $m(\mathscr{U})$ converges to an element x of X. By the lower semicontinuity of f, $r \ge \sup_{U \in \mathscr{U}} \inf_{y' \in U} f(m(y'), y') \ge f(x, y) \ge \inf_X f(y)$, a contradiction. □

By a similar argument, one also gets a sequential variant of the above result:

THEOREM 3.2. *Suppose that topologies on* X *and* Y *are first-countable and that* f *is lower semicontinuous on* $X \times \{y\}$. *If for each small* $\varepsilon > 0$, $^{\varepsilon}\mathrm{Min}_X f$ *is sequentially hypocompactoid at* y, *then* $\inf_X f$ *is lower semicontinuous at* y.

4. BOUNDS FOR CONTINGENT epi-DERIVATIVES OF MARGINAL FUNCTIONS

Consider an extended-real-valued function f on a product of linear spaces $X \times Y$.

THEOREM 4.1. *Let* Y *be equipped with a quasi-linear topology and* X *with an arbitrary topology. Let* $\inf_X f(y)$ *be finite. Then, for* $\alpha = -$ *or* $+$, *for each* $x \in \mathrm{Min}_X f(y)$,

$$D_{(\alpha;-)} \inf_X f(y) k \le \inf_{h \in X} D_{(\alpha;-)} f(x,y)(h,k). \tag{4.1}$$

Proof. Denote by σ the topology of Y and by o the chaotic topology of X. If x belongs to $\mathrm{Min}_X f(y)$, then

$$D_{(\alpha;\sigma^-)}\inf_X f(y)k = \lim_{\Gamma(t \overset{\alpha}{\to} 0,\, k' \overset{\sigma^-}{\to} k)} \frac{1}{t} \inf_{h' \in X} \left[f(x+th', y+tk') - f(x,y) \right].$$

We may carry $\inf_{h' \in X}$ before $1/t$, obtaining, for arbitrary $h \in X$,

$$D_{(\alpha;\sigma^-)}\inf_X f(y)k = \lim_{\Gamma(t \overset{\alpha}{\to} 0,\, k' \overset{\sigma^-}{\to} k,\, h' \overset{o^-}{\to} h)} \frac{1}{t} \left[f(x+th', y+tk') - f(x,y) \right].$$

The right-hand side in the above formula is equal to $D_{(\alpha;\, o \times \sigma^-)} f(x,y)(h,k)$ and, since the chaotic topology o is the coarsest among all topologies on X, the proof is complete. □

Remark 4.2. If X is a set without any structure one has, analogously to (4.1),

$$D_{(\alpha;\,-)}\inf_X f(y)k \le \inf_{x \in \mathrm{Min}_X f(y)} \left[D_{(\alpha;\,-)} f(x,\cdot) \right](y)k. \qquad (4.2)$$

This formula generalizes a half of [32, Lemma 1.8].

The assumption that $x \in \mathrm{Min}_X f(y)$ in both Theorem 2.1 and in (2.6) is essential. Indeed,

EXAMPLE 4.3. Define $f\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} \dfrac{y^4}{x^2+y^2}, & \text{if } x^2+y^2>0 \\ 0, & \text{otherwise}. \end{cases}$$

Then $\inf_X f$ is constantly equal to zero, while both epi-derivatives of f are equal to the usual (Fréchet) derivative. One has $Df(0,1)(0,-1) = -2$.

Consider now an extended-real-valued function g on X and a relation $A \subset X \times Y$. The linear spaces X and Y are equipped with quasi-linear topologies τ and σ. The contingent epi-derivative $D_{(-;\,\tau \times \sigma^-)}(g \dotplus \psi_A)$ may be estimated in many ways by various unilateral derivatives of g restricted to various approximating cones of A. In what follows, the symbols $\alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3$ take values either in $\{-1, +1\}$ or in $\{-, +\}$ (depending on the context).

PROPOSITION 4.4. *Let* $(x,y) \in A$. *If*

$$\alpha_1 + \beta_1 \ge 0, \qquad \alpha_2 + \beta_2 \ge 0, \qquad (4.3)$$

then

$$D_{(-;\tau\times\sigma^-)}(g \dotplus \psi_A)(x,y) \le D_{(\alpha_1;\tau^{\alpha_2})}g(x) \dotplus \psi_{T_{(\beta_1;\tau^{\beta_2},\sigma^{\beta_3})}A(x,y)}. \tag{4.4}$$

Proof. If $f\colon Z \to \overline{\mathbb{R}}$ and $B \subset Z$, where Z is a linear space then, for each $t > 0$, $h \in Z$, $z \in B$, we have

$$\begin{aligned}\frac{1}{t}&((f \dotplus \psi_B)(z+th) - (f \dotplus \psi_B)(z))\\ &= \left[\frac{1}{t}(f(z+th) - f(z))\right] \dotplus \psi_{(1/t)(B-z)}(h). \end{aligned}\tag{4.5}$$

The use of [9, Theorem 1.2], to estimate the $\Gamma(-,-,-)$-functional of the difference quotient of $g \dotplus \psi_A$ yields (4.4). $\square$

This proposition together with Theorem 4.1 gives

COROLLARY 4.5. *Let* $\inf_A g(y)$ *be finite. Then*

$$D_{(-;\sigma^-)}\inf_A g(y)k \le \inf_{x\in \mathrm{Min}_A g(y)} \inf_{h\in[T_{(\beta_1;\tau^{\beta_2},\sigma^{\beta_3})}A(x,y)]k} D_{(\alpha_1;\tau^{\alpha_2})}g(x,y)(h,k). \tag{4.6}$$

We shall give now sufficient conditions under which, for contingent derivatives, the opposite inequality in (4.1) holds as well.

THEOREM 4.6. *Let* $f(x,y) = \inf_X f(y)$ *be finite. Suppose that, for every small* $\varepsilon > 0$, *the relation*

$$(t',k') \mapsto \frac{1}{t'}\left[{}^{\varepsilon t'}\mathrm{Min}_X f(y+t'k') - x\right] \tag{4.7}$$

is hypocompactoid at $(0,k)$. *Then*

$$D_{(-;-)}\inf_X f(y)k \ge \inf_{h\in X} D_{(-;-)}f(x,y)(h,k). \tag{4.8}$$

Proof. We define the following function $g\colon \mathbb{R}_+\times X\times Y \to \mathbb{R}$ by setting

$$g(t',h',k') = \frac{1}{t'}[f(x+t'h', y+t'k') - f(x,y)], \qquad \text{if } t' > 0$$

$$g(0,h',k') = D_{(-;-)}f(x,y)(h',k').$$

One verifies that, for $t' > 0$,

$$^{\varepsilon}\mathrm{Min}_X\, g(t',k') = \frac{1}{t'}\left[{}^{\varepsilon t'}\mathrm{Min}_X\, f(y+t'k') - x\right].$$

By Theorem 3.1 the marginal function on X of g is lower semicontinuous at $(0,k)$, that is,

$$\lim_{\Gamma(t' \xrightarrow{-} 0,\, k' \xrightarrow{-} k)} \inf_{h' \in X} g(t',h',k') \geq \inf_{h \in X} g(0,h,k). \tag{4.9}$$

Since the topology on Y is quasi-linear and since $x \in \mathrm{Min}_X f(y)$, the left-hand side of (4.9) becomes $D_{(-;\,-)}\mathrm{inf}_X f(y)k$. We conclude that (4.9) is another form of (4.8). □

THEOREM 4.7. *Suppose that the topologies are first-countable. If for each* $\{\delta_n, t_n, k_n\}_{n \in \mathbb{N}}$ *convergent to* $(0,0,k)$ *there exist a subsequence* $\{n_m\}_{m \in \mathbb{N}}$ *of* $\{n\}_{n \in \mathbb{N}}$ *and a convergent sequence* $\{h_m\}_{m \in \mathbb{N}}$ *such that*

$$h_m \in \frac{1}{t_{n_m}}\left[{}^{\delta_{n_m}}\mathrm{Min}_X\, f(y + t_{n_m}k_{n_m}) - x\right],$$

then (4.11) *holds.*

Proof. For $\varepsilon > 0$, set $\delta_n = \varepsilon t_n$. Start the argument of the preceding proof and then apply Theorem 3.2. □

In order to give a geometric interpretation of our hypotheses we shall give this special case of Theorem 4.6:

COROLLARY 4.8. *Suppose that there exist* $t > 0$, *a neighborhood* Q *of* k, *and a selection* m *on* $y + (0,t)Q$ *of* $\mathrm{Min}_X f$ *such that*

$$\frac{1}{t'}[m(y+t'k') - x] \tag{4.10}$$

has nonempty adherence as $t' \to 0$ *and* $k' \to k$. *Then* (4.8) *holds.*

Proof. Observe that $(1/t')[m(y+t'k') - x]$ belongs to $(1/t')[\mathrm{Min}_X f(y+t'k') - x]$ and apply Theorem 4.6. □

The notion of adherence that is used in formulating the above corollary may be expressed in terms of limits of sets (2.1). The adherence of (4.10) is equal to

$$\mathrm{Lim}_{(t' \xrightarrow{-} 0,\, k' \xrightarrow{-} k,\, \tau^-)} \frac{1}{t'}[m(y+t'k') - x].$$

In particular, the assumption of Corollary 4.8 is satisfied, if $\lim_{t' \to 0, k' \to 0}(1/t')[m(y + t'k') - m(y)]$ exists for some selection of $\operatorname{Min}_X f$ on $y + [0, t]Q$ for which $m(y) = x$.

It follows from Theorem 4.6, that if there exists $x \in \operatorname{Min}_X f(y)$ such that for each $\varepsilon > 0$ the relation defined by (4.7) is hypocompactoid at $(0, k)$, then

$$D_{(-;\,-)}\inf_X f(y)k \geq \min_{x \in \operatorname{Min}_X f(y)} \inf_{h \in X} D_{(-;\,-)}f(x, y)(h, k). \quad (4.11)$$

We shall consider a weaker assumption of bounded hypocompactoidness, but we shall require it to hold for each $x \in \operatorname{Min}_X f(y)$ and we shall reinforce it by some additional hypotheses. In doing so, we shall extend Theorem 5.11 of Penot [32].

We call a relation M from a set Y to a linear topological space X *boundedly compactoid* along a filter $\mathscr{F}$ on Y, if for every bounded subset B of X, the relation $M \cap B$ is compactoid along $\mathscr{F}$. When Y is topologized, one defines analogously relations boundedly compactoid at y.

According to the definition, every relation valued in the topological dual E' (of a barreled space E) equipped with the weak topology $\sigma(E', E)$ (in particular, in finite dimension) is boundedly compactoid. For instance, as easily seen, if the relation (of implicit constraints): $\{y', x'\colon f(x', y') < +\infty\}$ is *B-tangentially compact* at (y, x) *in the direction* k in the sense of Penot [32], then for each $\varepsilon > 0$, the relation given by (4.7) is boundedly compactoid at $(0, k)$. Consider now the relation:

$$(\varepsilon', t', k') \mapsto {}^{\varepsilon'}\operatorname{Min}_X f(y + t'k'). \quad (4.12)$$

If we assume that the function f is *well set* at y [32, Definition 5.10] and X, Y are normed, then the relation given by (4.12) is hypocompactoid as $(0, 0, k)$. Let τ, σ be locally convex linear topologies on X and Y, respectively. Let $\inf_X f(y)$ be finite.

THEOREM 4.9. *Suppose that*

(i) *the relation* (4.12) *is hypocompactoid as* t', ε' *tend to* 0 *and* k' *tends to* k *in* σ;

(ii) *for each* $x \in \operatorname{Min}_X f(y)$ *for each* $\varepsilon > 0$, *the relation* (4.7) *is boundedly compactoid as* t' *tends to* 0 *and* k' *tends either to* k *or to* 0;

(iii) *for each* $x \in \operatorname{Min}_X f(y)$ *and every* $h \neq 0$, $D_{(-;\,-)}f(x, y)(h, 0) > 0$;

(iv) $\operatorname{Lim}_{(t' \to 0, k' \to k, \tau^-)} {}^{\varepsilon t'}\operatorname{Min}_X f(y + t'k') \subset \operatorname{Min}_X f(y)$.

Then, the estimate (4.11) *holds.*

Proof. It is known (e.g., [21]) that there is an ultrafilter $\mathscr{U}$ on $\mathbb{R}_+ \times Y$ convergent to $(0, k)$ such that

$$D_{(-;-)}\inf_X f(y)k = \lim_{\Gamma(\mathscr{U}^-)} \frac{1}{t'}\left[\inf_X f(y + t'k') - \inf_X f(y)\right]. \quad (4.13)$$

Consider the case where $\inf_X f(y + t'k') > -\infty$ for (t', k') in some $U \in \mathscr{U}$. By (i), for $\varepsilon > 0$, there is an $\mathscr{U}$-eventual selection m of the relation $(t', k') \mapsto {}^{\varepsilon t'}\mathrm{Min}_X f(y + t'k')$ convergent to an element x of $\mathrm{Min}_X f(y)$ in view of (iv). Then (4.13) is equal to

$$\lim_{\Gamma(\mathscr{U}^-)} \frac{1}{t'}\left[f(m(t', k'), y + t'k') - f(x, y)\right]. \quad (4.14)$$

If there is $U \in \mathscr{U}$ such that $\{(1/t')[m(t', k') - x]: (t', k') \in U\}$ is bounded, then by (ii) there is $h \in X$ such that (4.14) is greater or equal to $D_{(-;-)}f(x, y)(h, k)$ and (4.11) is proved.

The opposite case occurs when there are a continuous seminorm p and $U \in \mathscr{U}$ such that $\{r(t', k')/t', (t', k') \in U\}$ is an unbounded subset of $\mathbb{R}_+$, where $r(t', k') = p(m(t', k') - x) = r'$. Then (4.14) becomes

$$\lim_{\Gamma(\mathscr{U}^-)}\left(\frac{r'}{t'}\right)\frac{1}{r'}\left(f\left(x + \frac{r'(m(t', k') - x)}{r'}, y + r'\left(\frac{t'k'}{r'}\right)\right) - f(x, y)\right). \quad (4.15)$$

In view of (ii) (second variant), there is a limit h of $(m(t', k') - x)/r'$ (different from 0!) and thus (4.13) becomes $+\infty$, since $D_{(-;-)}f(x, y)(h, 0) > 0$. In this case (4.11) clearly holds.

We omit a similar proof of the complementary case in which $\inf_X f(y + t'k') = -\infty$ for (t', k') in some $U \in \mathscr{U}$. □

The above proof is similar to the original proof of Penot, but is not sequential. This feature broadens considerably the applicability; as we have already mentioned, assumption (ii) may be dropped if we deal with the weak topology of the dual of a barreled space. Besides we require bounded compactoidness of a relation that is always smaller and frequently much smaller than that of implicit constraints used in [32].

Condition (iv) is verified if, for instance, f is lower semicontinuous and $\inf_X f$ is upper semicontinuous at y (which is a very mild assumption!) [12].

Theorem 4.9 easily applies to minimization problems with explicit constraints. It hinges on the estimate

$$D_{(-;\tau\times\sigma^-)}(g \dotplus \psi_A)(x, y) \geq D_{(-;\tau^-)}g(x) \dotplus \psi_{T_{(-;\tau\times\sigma^-)}A(x, y)}. \quad (4.16)$$

The above formula is a consequence of [9, Theorem 1.1]. That theorem, however, may not be applied directly, $\dotplus$ not being a meet semi-homomorphism of $\overline{\mathbb{R}}$. The difficulty is avoided by observing that $g \dotplus \psi_A = g \vee \phi_A$ (where ϕ_A is equal $-\infty$ on A and $+\infty$ outside A) and that, analogously to 4.5,

$$\frac{1}{t}((f \vee \phi_B(z+th) - (f \vee \phi_B)(z))$$
$$= \frac{1}{t}(f(z+th) - f(z)) \vee \phi_{(1/t)(B-z)} \qquad (4.17)$$

if $z \in B$ and $f(z)$ is finite. Therefore, we have

COROLLARY 4.10. *Suppose that*

(i) *either the relation* $(t', k') \mapsto A(t + t'k')$ *or, for each* $r \in \mathbb{R}$, *the relation* $(t', k') \mapsto \{x'\colon g(x, y + t'k')\}$ *is hypocompactoid as* t' *tends to* 0 *and* k' *to* k;

(ii) *for each* $x \in \operatorname{Min}_A g(y)$ *the relation* $(t', k') \mapsto (1/t')(A(y + t'k') - x)$ *is boundedly compactoid as* t' *tends to* 0 *and* k' *tends either to* k *or to* 0;

(iii) *for each* $x \in \operatorname{Min}_A g(y)$ *and, for every* $h \neq 0$ *in* $(T_{(-;-)}A(x, y))0$, $D_{(-;-)}g(x)h > 0$;

(iv) (*of Theorem* 4.9) *holds for* $f = g \dotplus \psi_A$.

Then,

$$D_{(-;-)}\inf_A g(y)k \geq \min_{x \in \operatorname{Min}_X f(y)} \inf_{h \in (T_{(-;-)}A(x,y))k} D_{(-;-)}g(x)h. \qquad (4.18)$$

5. UPPER BOUNDED FOR EQUI-TANGENT epi-DERIVATIVES OF MARGINAL FUNCTIONS

Let τ, ξ be topologies on X, σ, θ topologies on Y, f an extended real-valued function on $X \times Y$ for which $\inf_X f(y)$ is finite.

We denote by $(\xi \times \theta)_{\operatorname{epi} f}$ the supremum of $\xi \times \theta$ and of the coarsest topology for which f is upper semicontinuous and, analogously by $\theta_{\operatorname{epi}(\inf_X f)}$ the supremum of θ and of the coarsest topology for which the marginal function $\inf_X f$ is upper semicontinuous.

THEOREM 5.1. *Let f be lower semicontinuous in $\xi \times \theta$. Let* $\operatorname{Min}_X f$ *be hypocompactoid at y from $\theta_{\operatorname{epi}(\inf_X f)}$ to ξ. Then for each $k \in Y$,*

$$D_{(+,\theta^+;\sigma^-)}\inf_X f(y)k \leq \sup_{x \in \operatorname{Min}_X f(y)} \operatorname{cl}_\sigma \inf_{h \in X} D_{(+,\xi\times\theta^+;\tau\times\sigma^-)} f(x, y)(h, k). \qquad (5.1)$$

Proof. By virtue of Theorem 3.1, the function $\inf_X f$ is lower semicontinuous at y with respect to $\theta_{\mathrm{epi}(\inf_X f)}$. One has that

$$\begin{aligned} D_{(+,\theta^+;\sigma^-)} \inf_X f(y)k &= D_{(+,\theta^+_{\mathrm{epi}(\inf_X f)};\,\sigma^-)} \inf_X f(y)k \\ &= \lim_{\Gamma(t' \xrightarrow{+} 0,\, y' \xrightarrow{\theta^+_{\mathrm{epi}(\inf_X f)}} y,\, k' \xrightarrow{\sigma^-} k)} \\ &\quad \frac{1}{t'}\left[\inf_X f(y'+t'k') - \inf_X f(y')\right]. \end{aligned}$$

It is known from [21] that

$$\lim_{\Gamma(+,\mathscr{N}(y)^+,-)} = \sup_{\mathscr{U}\in\beta\mathscr{N}(y)} \lim_{\Gamma(+,\mathscr{U}^+,-)}, \tag{5.2}$$

where $\beta\mathscr{N}(y)$ denotes the set of all ultrafilters finer than the filter $\mathscr{N}(y)$. By hypocompactoidness, for each $\mathscr{U}\in\beta\mathscr{N}(y)$ there exists a $\mathscr{U}$-eventual selection $m_{\mathscr{U}}$ of $\mathrm{Min}_X f$ and x in X such that $x\in\lim_\xi m_{\mathscr{U}}(\mathscr{U})$. In particular, there exists $U\in\mathscr{U}$ such that for each $y'\in U$

$$\begin{aligned} &\frac{1}{t'}\left[\inf_X f(y'+t'k') - \inf_X f(y')\right] \\ &\quad = \inf_{h'\in X} \frac{1}{t'}\left[f(m_{\mathscr{U}}(y')+t'h',\, y'+t'k') - f(m_{\mathscr{U}}(y'),y')\right] \end{aligned}$$

and, thus, recalling that o stands for the chaotic topology, we have that

$$\begin{aligned} &\lim_{\Gamma(t\xrightarrow{+}0,\,\mathscr{U}^+,\sigma^-)} \inf_{h'\in X} \frac{1}{t'}\left[f(m_{\mathscr{U}}(y')+t'h',\, y'+t'k') - f(m_{\mathscr{U}}(y'),y'\right] \\ &\quad = \lim_{\Gamma(t\xrightarrow{+}0,\,(m_{\mathscr{U}}\times 1)\mathscr{U}^+,\, o\times\sigma^-)} \frac{1}{t'}\left[f(x'+t'h',\, y'+t'k') - f(x',y')\right], \end{aligned} \tag{5.3}$$

where $\mathbf{1}$ denotes the identity on Y. The expression in (5.3) is σ-l.s.c. as a function of k. We have that $(m_{\mathscr{U}}\times \mathbf{1})\mathscr{U}\supset \mathscr{N}_\xi(x)\times\mathscr{N}_{\theta_{\mathrm{epi}(\inf_X f)}}(y)$. We shall show that $(m_{\mathscr{U}}\times\mathbf{1})\mathscr{U}$ converges to (x,y) in $(\xi\times\theta)_{\mathrm{epi}\, f}$ and that $x\in \mathrm{Min}_X f(y)$.

Since f is lower semicontinuous with respect to $\xi\times\theta$ (thus with respect to $\xi\times\theta_{\mathrm{epi}(\inf_X f)}$) and $\inf_X f$ is upper semicontinuous with respect to $\theta_{\mathrm{epi}(\inf_X f)}$, we have, by virtue of Corollary 2.2 of [12], that

$$x\in\lim_\xi m_{\mathscr{U}}(\mathscr{U}) \subset \mathrm{Lim}_{(\mathscr{U}^-,\xi^-)} \mathrm{Min}_X f \subset \mathrm{Min}_X f(y).$$

Now if $r > f(x,y) = \inf_X f(y)$, then there is $U\in\mathscr{U}$ such that for $y'\in U$, $\inf_X f(y') < r$, hence $(m_{\mathscr{U}}\times\mathbf{1})U\subset\{(x',y')\colon f(x',y')<r\}$.

Therefore (5.3) is less than $\mathrm{cl}_\sigma(\inf_{h\in X} D_{(+,\xi\times\theta^+;\tau\times\sigma^-)} f(x,y))(h,k)$ for an arbitrary topology τ on X. In view of (5.2) the proof is complete. □

Remark 5.2. In the course of demonstration we have actually proved that $\mathrm{Min}_X f \times \mathbf{1}$ is $(\xi \times \theta)_{\mathrm{epi}\, f}$-hypocompactoid $\theta_{\mathrm{epi}(\inf_X f)}$-locally at y. As we have observed in the introduction, the condition of tameness entails our hypocompactoidness (see [11]). Therefore the corresponding estimates of Rockafellar and of Hiriart-Urruty follow from Theorem 5.1.

We call a topology *regular*, if each neighborhood filter admits a base composed of closed sets (we do not require that the topology by Hausdorff).

COROLLARY 5.3. *Suppose that* ξ, θ *are regular. Let* f *be lower semicontinuous with respect to* $\xi \times \theta$. *Suppose that there exists a* $\mathscr{N}_{\theta_{\mathrm{epi}(\inf_X f)}}(y)$-*eventual* ξ-*compactoid selection* m *of* $\mathrm{Min}_X f$. *Then there exists a subset* K *of* $\mathrm{Min}_X f(y)$ *such that* $K \times \{y\}$ *is* $(\xi \times \theta)_{\mathrm{epi}\, f}$-*compact and*

$$D_{(+,\theta^+;\sigma^-)} \inf_X f(y) k \leq \sup_{x \in K} \inf_{h \in X} D_{(+,\xi\times\theta^+;\tau\times\sigma^-)} f(x, y)(h, k). \quad (5.4)$$

Proof. By Remark 5.2, for each ultrafilter $\mathscr{U}$ finer than $\mathscr{N}_{\theta_{\mathrm{epi}(\inf_X f)}}(y)$ there is an $x \in X$ such that $(m \times \mathbf{1})\mathscr{U}$ converges to (x, y) in $(\xi \times \theta)_{\mathrm{epi}\, f}$. The set

$$K = \left[\mathrm{Lim}_{(\theta_{\mathrm{epi}(\inf_X f)}, \mathscr{N}(\xi\times\theta)_{\mathrm{epi}\, f}(y)^-)} m\right](y) \quad (5.5)$$

is a subset of $\mathrm{Min}_X f(y)$ for which (5.4) holds.

We shall show that $K \times \{y\}$ is compact in $(\xi \times \theta)_{\mathrm{epi}\, f}$. For (each function) f the coarsest topology for which f is continuous is regular (in our sense). Hence $(\xi \times \theta)_f$ (the supremum of that topology and of $\xi \times \theta$) is regular and as f is supposed to be lower $\xi \times \theta$-semicontinuous, $(\xi \times \theta)_{\mathrm{epi}\, f} = (\xi \times \theta)_f$.

By virtue of [13, Corollary 4.13] (see also [30]) the limit in (5.5) is $(\xi \times \theta)_{\mathrm{epi}\, f}$-compact, so is $K \times \{y\}$. □

If the selection used in Corollary 5.3 is not only compactoid at y but continuous at y and if ξ, θ are Hausdorff, then K in (5.4) reduces to one point (see [23, (3.3′)]).

Rockafellar defines in [38] directionally Lipschitzian functions. His definition is extended by the following one: a function $f\colon X \to \overline{\mathbb{R}}$ is τ-*directionally* σ-*Lipschitzian* at x towards h, if $D_{(+,\sigma^+;\tau^+)} f(x)h < +\infty$.

COROLLARY 5.4. *Let* $\tau \geq \xi$ *be linear Hausdorff topologies on* X, $\sigma \geq \theta$ *linear Hausdorff topologies on* Y. *Let* k *be such that for each* $x \in \mathrm{Min}_X f(y)$ *and each* $h \in X$, f *is* $\tau \times \sigma$-*directionally* $\xi \times \theta$-*Lipschitzian at* (x, y) *towards* (h, k). *If there exists a* ξ-*compactoid* $\theta_{\mathrm{epi}(\inf_X f)}$-*local selection of* $\mathrm{Min}_X f$ *at* y, *then there is* $x \in \mathrm{Min}_X f(y)$ *such that*

$$D_{(+,\theta^+;\sigma^-)} \inf_X f(y) k \leq \inf_{h \in X} D_{(+,\xi\times\theta^+;\tau\times\sigma^-)} f(x, y)(h, k). \quad (5.6)$$

Proof. In view of [38, Theorem 3], the directional Lipschitz condition implies that

$$D_{(+,\xi\times\theta^+;\iota\times\iota^-)}f(x,y)(h,k) = D_{(+,\xi\times\theta^+;\tau\times\sigma^-)}f(x,y)(h,k)$$

for each $h \in X$ and $x \in \operatorname{Min}_X f(y)$. On the other hand, by (2.16), the functions $(x, y) \mapsto D_{(+,\xi\times\theta^+;\iota\times\iota^-)}f(x, y)(h, k)$ are upper semicontinuous with respect to $(\xi \times \theta)_{\operatorname{epi} f}$ as is their infimum as h ranges over X. Therefore the supremum in (5.4) is attained for each k. □

COROLLARY 5.5. *Under the hypotheses of Corollary* 5.4 (*valid for each $k \in Y$ separately*),

$$\operatorname{dom}(D_{(+,\theta^+;\sigma^-)}\inf_X f(y)) \supset \bigcap_{x\in\operatorname{Min}_X f(y)} \operatorname{dom}\Big(\inf_{h\in X} D_{(+,\xi\times\theta^+;\tau\times\sigma^-)}f(x,y)(h,\cdot)\Big). \quad (5.7)$$

If f is directionally Lipschitzian at (x, y) towards (h, k) for each $h \in X$ and each $x \in \operatorname{Min}_X f(y)$, the maximum is realized on (5.4) for every k.

Consider now the case of $f = g \dotplus \psi_A$, where $g: X \times Y \to \overline{\mathbb{R}}$ and $A \subset X \times Y$. This case, as seen before, corresponds to the minimization problems in which some constraints are explicitly given by A. It is known [37] that if there is (h, k) belonging to the equi-tangent cone $T_{(+,+;-)}A(x, y)$ towards which g is directionally Lipschitzian at (x, y), then

$$D_{(+,+;-)}(g \dotplus \psi_A)(x,y)(h,k) \le D_{(+,+;-)}g(x,y)(h,k)$$

for each (h, k) in $T_{(+,+;-)}A(x, y)$. This inequality enables us to deduce the following from Theorem 5.1.

COROLLARY 5.6. *Suppose that $\tau \ge \xi$, $\sigma \ge \theta$ are linear Hausdorff topologies, g is a lower $\xi \times \theta$-semicontinuous function on $X \times Y$, A is a relation from Y to X hypocompactoid at y from θ to ξ, satisfying* $(\operatorname{Lim}_{(\theta^-,\xi^-)} A)(x, y) \subset A(x, y)$ *and such that for each $x \in \operatorname{Min}_A g(y)$ there exists an $(h, k) \in T_{(+,\xi\times\theta^+;\tau\times\sigma^-)}A(x, y)$ towards which g is $\tau \times \sigma$-directionally $\xi \times \theta$-Lipschitzian at (x, y). Then*

$$D_{(+,+;-)} \inf_A g(y)k \le \sup_{x\in\operatorname{Min}_A g(y)} \inf_{h\in(T_{(+,+;-)}A(x,y))k} D_{(+,+;-)}g(x,y)(h,k). \quad (5.8)$$

6. Subgradient Inclusions

Let X, Y be linear spaces and $\langle\,,\rangle$: $X \times Y \to \mathbb{R}$ a bilinear coupling function that separates points. Then, there is a lattice isomorphism between the complete lattice of convex, $\sigma(Y, X)$-closed subsets of Y and that of sublinear, $\sigma(X, Y)$-lower semicontinuous proper functions g on X, enriched by the constant function $-\infty$, given by

$$\partial g = \{y \in Y : \langle x, y\rangle \le g(x), x \in X\},$$

the inverse isomorphism being defined by

$$A \mapsto \sup_{y \in A} \langle \cdot\,, y\rangle.$$

Hence, for a family $\{g_i\}_{i \in I}$ of such functionals,

$$\partial\Big(\sup_{i \in I} g_i\Big) = \operatorname{cl} \operatorname{co}\Big(\bigcup_{i \in I} \partial g_i\Big) \quad \text{and} \quad \partial\Big(\inf_{i \in I} g_i\Big) = \bigcap_{i \in I} \partial g_i, \tag{6.1}$$

where cl co stands for the closed convex hull.

A dual lattice isomorphism exists between the complete lattice of convex $\sigma(Y, X)$-closed cones in Y with vertex at zero and that of convex, $\sigma(X, Y)$-closed cones in X with vertex at zero and is given by

$$A^\circ = \{x : \langle x, y\rangle \le 0 \text{ for } y \in A\}.$$

Accordingly, for a family $\{A_i\}_{i \in I}$ of such cones

$$\Big(\bigcup_{i \in I} A_i\Big)^\circ = \bigcap_{i \in I} A_i^\circ \quad \text{and} \quad \Big(\bigcap A_i\Big)^\circ = \operatorname{cl} \operatorname{co} \bigcup_{i \in I} A_i^\circ. \tag{6.2}$$

Proposition 6.1 [43, Proposition 4]. *If p is a proper sublinear lower semicontinuous function on a locally convex space X and if K is closed convex cone in X (with vertex at 0), then,*

$$\partial(p + \psi_K) = \operatorname{cl}(\partial p + K^\circ). \tag{6.3}$$

Therefore for sublinear lower semicontinuous functions p and q such that p is proper and q is not (consequently q admits only the values $+\infty$ and $-\infty$), one has

$$\partial \sup(p,q) = \mathrm{cl}(\partial p + (\mathrm{dom}\, q)^{\circ}). \tag{6.4}$$

An equi-tangent derivative $D_{(+,\xi^+,\tau^-)}f(x)$ (for ξ a linear topology, τ a linear locally convex topology such that $\tau \geq \xi$) is either a sublinear lower τ-semicontinuous proper function, or is equal to $-\infty$ on its effective domain dom $D_{(+,+;-)}f(x) = \{h: D_{(+,\xi^+;\tau^-)}f(x)h < +\infty\}$ which is a convex cone with vertex at zero. In what follows, after Rockafellar, we denote by

$$\partial f(x) = \partial D_{(+,+;-)}f(x), \qquad \partial^{\infty} f(x) = \left[\mathrm{dom}\, D_{(+,+;-)}f(x)\right]^{\circ}$$

respectively the *subdifferential* and the *singular subdifferential* of f at x. Of course, if $\partial f(x)$ is empty, then $D_{(+,+;-)}f(x)h$ equals $-\infty$ on its domain.

Now let τ, σ be locally convex topologies on X and Y, respectively, and let X', Y' stand for their topological duals. Let ξ and θ be other locally convex topologies on X and Y. Then, for a function f: $X \times Y \to \overline{\mathbb{R}}$, $\partial f(x,y)$ and $\partial^{\infty} f(x,y)$ are understood as subdifferentials relative to $D_{(+,\xi\times\theta^+;\tau\times\sigma^-)}f(x,y)$, hence they are subsets of $X' \times Y'$ (and thus relations from X' to Y'). Following Rockafellar [35], we set

$$M(y) = \bigcup_{x\in \mathrm{Min}_X f(y)} [\partial f(x,y)]0 \tag{6.5}$$

$$M^{\infty}(y) = \bigcup_{x\in \mathrm{Min}_X f(y)} [\partial^{\infty} f(x,y)]0, \tag{6.6}$$

where 0 stands for the origin of X.

THEOREM 6.2. *For locally convex topologies such that $\tau \geq \xi$, $\sigma \geq \theta$, if (X,τ) is a reflexive Banach space and if the hypotheses of Theorem* 5.1 *hold, then*

$$\partial \inf_X f(y) \subset \mathrm{cl}_{\sigma}\, \mathrm{co}(M(y) + M^{\infty}(y)). \tag{6.7}$$

Proof. Since $\tau \geq \xi$ and $\sigma \geq \theta$, $D_{(+,\xi\times\theta^+;\tau\times\sigma^-)}f(x,y)$ is sublinear for each x, y. Consequently, the functions

$$a_{x,y}(k) = \mathrm{cl}_{\sigma} \inf_{h\in X} D_{(+,\xi\times\theta^+;\tau\times\sigma^-)}f(x,y)(k,h)$$

are sublinear. Let $I(y)$ (resp. $I^\infty(y)$) be the subset of $\mathrm{Min}_X f(y)$ such that for $x \in I(y)$ (resp. $x \in I^\infty(y)$) the corresponding function is proper (is not proper). Then, by (5.1), in view of (6.1) and (6.4)

$$\partial \inf_X f(y) \subset \mathrm{cl}_\sigma\left(\mathrm{cl}_\sigma \mathrm{co} \bigcup_{x \in I(y)} \partial a_{xy} + \mathrm{cl}_\sigma \mathrm{co} \bigcup_{x \in I^\infty(y)} (\mathrm{dom}\, a_{xy})^\circ\right)$$

$$\subset \mathrm{cl}_\sigma \mathrm{co}\left(\bigcup_{x \in I(y)} \partial a_{xy} + \bigcup_{x \in I^\infty(y)} (\mathrm{dom}\, a_{xy})^\circ\right). \tag{6.8}$$

Now, $z \in \partial a_{xy}$ (resp. $z \in (\mathrm{dom}\, a_{xy})^\circ$), implies that $(0, z) \in \partial f(x, y)$ (resp. $(0, z) \in \partial^\infty f(x, y)$). According to (6.5), (6.6), formula (6.8) yields (6.7). □

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